

On graphs with $m(\partial_1^L) = n - 3$ *

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Abstract Let $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L$ be the distance Laplacian eigenvalues of a connected graph G and $m(\partial_i^L)$ the multiplicity of ∂_i^L . It is well known that the graphs with $m(\partial_1^L) = n-1$ are complete graphs. Recently, the graphs with $m(\partial_1^L) = n-2$ have been characterized by Celso et al. In this paper, we completely determine the graphs with $m(\partial_1^L) = n-3$.

Keywords: Distance Laplacian matrix; Laplacian matrix; Largest eigenvalue; Determined by distance Laplacian spectrum

AMS subject classifications: 05C50; 05C12; 15A18

1 Introduction

In this paper we only consider simple connected graphs. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. The *distance* between v_i and v_j , denoted by $d_G(v_i, v_j)$, is defined as the length of a shortest path between them. The *diameter* of G , denoted by $d(G)$, is the maximum distance between any two vertices of G . The *distance matrix* of G , denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i, j) -entry is equal to $d_G(v_i, v_j)$, $i, j = 1, 2, \dots, n$. The *transmission* $Tr(v_i)$ of a vertex v_i is defined as the sum of the distances between v_i and all other vertices in G , that is, $Tr(v_i) = \sum_{j=1}^n d_G(v_i, v_j)$. For more details about the distance matrix we refer the readers to [1]. Aouchiche and Hansen [2] introduced the Laplacian for the distance matrix of G as $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$, where $Tr(G) = \text{diag}(Tr(v_1), Tr(v_2), \dots, Tr(v_n))$ is the diagonal matrix of the vertex transmissions in G . The eigenvalues of $\mathcal{D}^L(G)$, listed by $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$, are called the *distance Laplacian eigenvalues* of G . The multiplicity of ∂_i^L is denoted by $m(\partial_i^L)$. The distance eigenvalues together with their multiplicities is called the *distance Laplacian spectrum* of G , denoted by $\text{Spec}_L(G)$.

The distance Laplacian matrix aroused many active studies, such as [1, 6, 10, 11]. Graphs with few distinct eigenvalues form an interesting class of graphs and possess nice combinatorial properties. With respect to distance Laplacian eigenvalues, we focus on the graphs with $m(\partial_1^L)$ being large. Denote by $\mathcal{G}(n)$ the set of connected graphs of order n . Let $\mathcal{G}(n, k) = \{G \in \mathcal{G}(n) \mid m(\partial_1^L) = k\}$ be the set of connected graphs with $m(\partial_1^L) = k$.

*Supported by the National Natural Science Foundation of China (Grant Nos. 11671344, 11531011, 11626205).

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Aouchiche and Hansen [1] proved that $\mathcal{G}(n, n-1) = \{K_n\}$ and conjectured that $\mathcal{G}(n, n-2) = \{K_{1,n-1}, K_{n/2, n/2}\}$, which has been confirmed by Celso et al. [6]. Motivated by their work, we try to characterize $\mathcal{G}(n, n-3)$. In this paper, we completely determine the graphs in $\mathcal{G}(n, n-3)$ (Theorem 3.3). By the way, we show that all these graphs are determined by their distance Laplacian spectra (Corollary 3.3).

2 Preliminaries

Let G be a connected graph, we always denote by $N_G(v)$ the neighbor set of v in G , that is, $N_G(v) = \{u \in V(G) \mid u \sim v\}$. The i -th largest distance Laplacian eigenvalue of G is denoted by $\partial_i^L(G)$, whose multiplicity is denoted by $m(\partial_i^L(G))$. When it is clear from the context which graph G we mean, we delete G from the notations like $d_G(v_i, v_j)$, $N_G(v)$, $\partial_i^L(G)$ and $m(\partial_i^L(G))$. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S .

As usual, we always write, respectively, K_n , P_n and C_n for the complete graph, the path and the cycle on n vertices. For integers $a_1, a_2, \dots, a_k \geq 1$, let K_{a_1, a_2, \dots, a_k} denote the complete k -partite graph on $a_1 + a_2 + \dots + a_k$ vertices. Let G be a connected graph, denote by \bar{G} the *complement* of G , which is a graph with vertex set $V(\bar{G}) = V(G)$ and two vertices are adjacent whenever they are not adjacent in G . Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, the *(disjoint-)union* of G_1 and G_2 is the graph $G_1 \cup G_2$, whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$. The *join* of G_1 and G_2 is the graph $G_1 \nabla G_2$, which is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2 . Moreover, we write $mG = \underbrace{G \cup G \cup \dots \cup G}_m$ for an integer $m \geq 2$.

At first, we introduce the famous Cauchy interlacing theorem.

Theorem 2.1 ([8]). *Let A be a real symmetric matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ and let M be a principal submatrix of A with order $m \leq n$ and eigenvalues $\lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_m(M)$. Then $\lambda_i(A) \geq \lambda_i(M) \geq \lambda_{n-m+i}(A)$, for all $1 \leq i \leq m$.*

Let G be a graph on n vertices, denote by $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ the Laplacian eigenvalues of G and $m(\mu_i)$ the multiplicity of μ_i . There are many pretty properties for Laplacian eigenvalues.

Lemma 2.1 ([4]). *Let G be a graph on n vertices with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$. Then we have the following results.*

- (i) *Denote by $m(0)$ the multiplicity of 0 as a Laplacian eigenvalue and $w(G)$ the number of connected components of G . Then $w(G) = m(0)$.*
- (ii) *G has exactly two distinct Laplacian eigenvalues if and only if G is a union of complete graphs of the same order and isolate vertices.*
- (iii) *The Laplacian eigenvalues of \bar{G} are given by $\mu_i(\bar{G}) = n - \mu_{n-i}$ for $i = 1, 2, \dots, n-1$ and $\mu_n(\bar{G}) = 0$.*
- (iv) *Let H be a graph on m vertices with Laplacian eigenvalues $\mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_m = 0$, then the Laplacian spectrum of $G \nabla H$ is*

$$\{n+m, m+\mu_1, m+\mu_2, \dots, m+\mu_n, n+\mu'_1, n+\mu'_2, \dots, n+\mu'_m, 0\}.$$

With respect to distance Laplacian eigenvalues, there are some similar results. The following results are given by Aouchiche and Hansen.

Theorem 2.2 ([2]). *Let G be a connected graph on n vertices with $d(G) \leq 2$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian spectrum of G . Then the distance Laplacian spectrum of G is $2n - \mu_{n-1} \geq 2n - \mu_{n-2} \geq \dots \geq 2n - \mu_1 \geq \partial_n^L = 0$. Moreover, for every $i \in \{1, 2, \dots, n-1\}$ the eigenspaces corresponding to μ_i and to $2n - \mu_i$ are the same.*

Theorem 2.3 ([2]). *Let G be a connected graph on n vertices. Then $\partial_{n-1}^L \geq n$ and $\partial_{n-1}^L = n$ if and only if \bar{G} is disconnected. Furthermore, the multiplicity of n as a distance Laplacian eigenvalue is one less than the number of connected components of \bar{G} .*

Theorem 2.4 ([2]). *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph G' obtained from G by the deletion of an edge. Let $\partial_1^L, \partial_2^L, \dots, \partial_n^L$ and $\partial_1'^L, \partial_2'^L, \dots, \partial_n'^L$ denote the distance Laplacian spectra of G and G' respectively. Then $\partial_i'^L \geq \partial_i^L$ for all $i = 1, \dots, n$.*

A graph G is said to be a *cograph* if it contains no induced P_4 . There's a pretty result about cographs.

Lemma 2.2 ([5]). *Given a graph G , the following statements are equivalent:*

- 1) G is a cograph.
- 2) The complement of any connected subgraph of G with at least two vertices is disconnected.
- 3) Every connected induced subgraph of G has diameter less than or equal to 2.

3 Main results

Recall that $\mathcal{G}(n, k) = \{G \in \mathcal{G}(n) \mid m(\partial_1^L) = k\}$. Aouchiche and Hansen [1] proved that $\mathcal{G}(n, n-1) = \{K_n\}$. Recently, Celso et al. [6] proved that $\mathcal{G}(n, n-2) = \{K_{1,n-1}, K_{n/2, n/2}\}$. They also made efforts to characterize $\mathcal{G}(n, n-3)$. Though they did not give a complete characterization, their ideas are enlightening. Especially, they proved that the graphs in $\mathcal{G}(n, n-3)$ contain no induced P_5 .

Lemma 3.1 ([6], Theorem 4.1). *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$ then G does not contain induced P_5 .*

Remark 1. *If G does not contain induced P_5 , then $d(G) \leq 3$. Note that $K_n \notin \mathcal{G}(n, n-3)$. We obtain that $d(G) = 2$ or $d(G) = 3$ for any graph $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$.*

Lemma 3.2 ([6], Theorem 3.3). *If G is a connected graph then $\partial_1^L \geq \max_{v \in V(G)} \text{Tr}(v) + 1$ with equality holds if and only if $G \cong K_n$.*

Lemma 3.3. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then ∂_1^L is integral.*

Proof. Let $f(x)$ be the characteristic polynomial of $\mathcal{D}^L(G)$. As $\mathcal{D}^L(G)$ only contains integral entries, we obtain that $f(x)$ is a monic polynomial with integral coefficients. Let $p(x)$ be the minimal polynomial of ∂_1^L , then $p(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$ and $(p(x))^{n-3} \mid f(x)$. We assume that $p(x)$ is a polynomial of degree at least 2. Therefore, $p(x)$ has another root $\partial \neq 0$, which is also a distance Laplacian eigenvalue of G with multiplicity $n-3$. It leads to that $n \leq 2(n-3) \leq n-1$, a contradiction. Thus, we have $p(x) = x - \partial_1^L$ and the result follows. \square

From Lemmas 3.2 and 3.3, we get the following result.

Corollary 3.1. *Let $G = (V, E) \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then we have $\partial_1^L \geq \max_{v \in V} \text{Tr}(v) + 2$. Furthermore, if there exists $v_0 \in V$ such that $\partial_1^L = \text{Tr}(v_0) + 2$, then $\text{Tr}(v_0) = \max_{v \in V} \text{Tr}(v)$.*

Proof. Obviously, $G \neq K_n$. By Lemma 3.2, we have that $\partial_1^L > \max_{v \in V} \text{Tr}(v) + 1$. Besides, we get that ∂_1^L is integral from Lemma 3.3. Therefore, we have that $\partial_1^L \geq \max_{v \in V} \text{Tr}(v) + 2$. Furthermore, if $\partial_1^L = \text{Tr}(v_0) + 2$, then we have $\text{Tr}(v_0) + 2 \geq \max_{v \in V} \text{Tr}(v) + 2$. It follows that $\text{Tr}(v_0) = \max_{v \in V} \text{Tr}(v)$. \square

We say that a graph G is P_5 -free if it does not contain induced P_5 . From Lemma 3.1, all graphs in $\mathcal{G}(n, n-3)$ are P_5 -free. By Remark 1, a P_5 -free graph may have diameter 2 or 3. Now we discuss P_5 -free graphs with diameter 3.

Lemma 3.4. *Let G be a connected P_5 -free graph on $n \geq 5$ vertices with $d(G) = 3$. Then at least one of I_i for $i = 1, 2, 3, 4, 5$ (shown in Fig. 1) is an induced subgraph of G .*

Proof. Suppose that $d(v_1, v_4) = 3$ and $P = v_1 v_2 v_3 v_4$ is a shortest path from v_1 to v_4 . Since $n \geq 5$ and G is connected, there exists $u \in V(G) \setminus V(P)$ such that $N(u) \cap V(P) \neq \emptyset$, where $N(u) = \{v \in V(G) \mid v \sim u\}$ is the neighbour set of u in G . Moreover, since $d(v_1, v_4) = 3$, we have that v_1 and v_4 cannot be adjacent to u simultaneously, that is, $\{v_1, v_4\} \not\subseteq N(u)$. Therefore, we have $1 \leq |N(u) \cap V(P)| \leq 3$.

Assume that $|N(u) \cap V(P)| = 1$. We claim that $N(u) \cap V(P) = \{v_2\}$ or $\{v_3\}$ since G is P_5 -free. Both of them lead to the induced subgraph I_1 .

Assume that $|N(u) \cap V(P)| = 2$. We claim that $N(u) \cap V(P) = \{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_3\}, \{v_2, v_4\}$, or $\{v_2, v_3\}$ because $\{v_1, v_4\} \not\subseteq N(u)$. The former two cases lead to the induced subgraph I_2 , the next two cases lead to the induced subgraph I_3 and the last case leads to the induced subgraph I_4 .

Assume that $|N(u) \cap V(P)| = 3$. We claim that $N(u) \cap V(P) = \{v_1, v_2, v_3\}$ or $\{v_2, v_3, v_4\}$ because $\{v_1, v_4\} \not\subseteq N(u)$. Both cases lead to the induced subgraph I_5 . \square

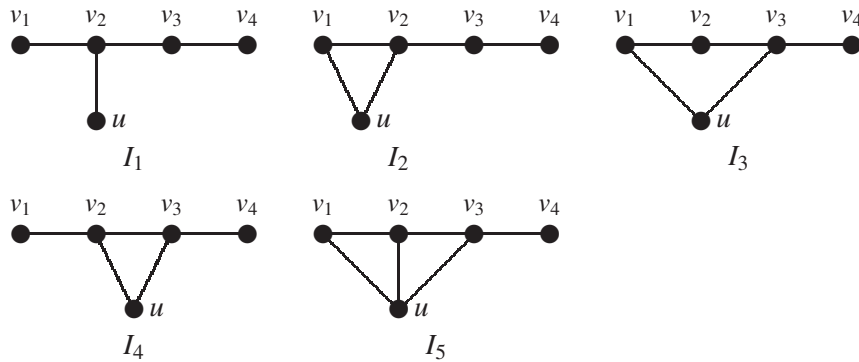


Figure 1: The graphs I_1, I_2, \dots, I_5

Next we introduce a tool which will be used frequently.

Lemma 3.5. Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$ and M a principal submatrix of $\mathcal{D}^L(G)$ of order 5. Then ∂_1^L is also an eigenvalue of M with multiplicity at least two. Furthermore, for each $1 \leq k \leq 5$, there exists an eigenvector $z = (z_1, z_2, \dots, z_5)^T$ of M with respect to ∂_1^L such that $z_k = 0$ and $\sum_{i=1}^5 z_i = 0$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_5$ be the eigenvalues of M . By Theorem 2.1, we have $\partial_1^L = \partial_{n-4}^L \leq \lambda_1 \leq \partial_1^L$ and $\partial_1^L = \partial_{n-3}^L \leq \lambda_2 \leq \partial_2^L = \partial_1^L$. Therefore, we have $\lambda_1 = \lambda_2 = \partial_1^L$. Suppose that $x = (x_1, \dots, x_5)^T$ and $y = (y_1, \dots, y_5)^T$ are two independent eigenvectors of M with respect to ∂_1^L . For each fixed integer $1 \leq k \leq 5$, by linear combination of x and y , we get the eigenvector $z = (z_1, \dots, z_5)^T$ satisfying $z_k = 0$. Let $z^* = (z_1, \dots, z_5, 0, \dots, 0)^T$. Note that $\partial_1^L \geq \frac{z^{*T} \mathcal{D}^L(G) z^*}{z^{*T} z^*} = \frac{z^T M z}{z^T z} = \partial_1^L$. We get that z^* is an eigenvector of $\mathcal{D}^L(G)$ with respect to $\partial_1^L \neq 0$. Note that the all-ones vector j is an eigenvector of $\mathcal{D}^L(G)$ with respect to 0. We have $z^{*T} j = \sum_{i=1}^5 z_i = 0$. \square

Denote by $J(a, b)$ the graph obtained from $K_{1,a} \cup K_{1,b}$ by joining each pendent vertex of $K_{1,a}$ with every pendent vertex of $K_{1,b}$ (shown in Fig. 2). The non-pendent vertices of $K_{1,a}$ and $K_{1,b}$ are called the *roots* of $J(a, b)$.

Lemma 3.6. Let G be a connected P_5 -free graph on $n \geq 5$ vertices with diameter $d(G) = 3$. If none of I_1, I_2, I_4 and I_5 is an induced subgraph of G , then $G = J(a, b)$ for some positive integers $a, b \geq 1$ and $a + b + 2 = n$.

Proof. Let $d(v_1, v_4) = 3$ and $P = v_1 v_2 v_3 v_4$ a shortest path between v_1 and v_4 . By Lemma 3.4, at least one of I_i (shown in Fig. 1) is an induced subgraph of G for $i = 1, 2, \dots, 5$. Since none of I_1, I_2, I_4 or I_5 is an induced subgraph of G , we obtain that G contains induced I_3 .

Note that $I_3 = J(2, 1)$ with roots v_1 and v_4 is an induced subgraph of G . We may assume that $G' = J(a, b)$ with roots v_1 and v_4 is the maximal induced subgraph of G including $J(2, 1)$. Denote by $U_1 = N_{G'}(v_1) = \{x_1, x_2, \dots, x_a\}$ and $U_2 = N_{G'}(v_4) = \{y_1, y_2, \dots, y_b\}$. Obviously, $v_2, v_3 \in U_1$ and $v_3 \in U_2$. In what follows we will show that $G = J(a, b)$ with roots v_1 and v_4 .

By the way of contradiction, assume that $G \neq J(a, b)$. Then there exists $v \in V(G) \setminus V(G')$ such that $N_G(v) \cap V(G') \neq \emptyset$. Since $d(v_1, v_4) = 3$, v is adjacent to at most one of v_1 and v_4 . We claim that v is exactly adjacent to one of v_1 and v_4 . Otherwise, we have $v \not\sim v_1, v_4$. Then $N_G(v) \cap U_1 \neq \emptyset$ or $N_G(v) \cap U_2 \neq \emptyset$. If v is adjacent to some vertex in $N_G(v) \cap U_1$ and some vertex in $N_G(v) \cap U_2$, say $v \sim x_1$ and $v \sim y_1$ (see Fig. 2 (1)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_4$, a contradiction. If v is only adjacent to some vertex in $N_G(v) \cap U_1$, say $v \sim x_1$ (see Fig. 2 (2)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_1$, a contradiction. If v is only adjacent to some vertex in $N_G(v) \cap U_2$, say $v \sim y_1$, then we also get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_1$, a contradiction. Now we need to consider the following two situations.

Case 1. $v \sim v_1$ and $v \not\sim v_4$;

First, we will show that $U_2 \subseteq N_G(v)$. Otherwise, there exists some vertex in U_2 not adjacent to v , say $v \not\sim y_1$. Now, if $v \not\sim x_1$ (see Fig. 2 (3)), then we get the induced subgraph $G[v, v_1, x_1, y_1, v_4] = P_5$, a contradiction; if $v \sim x_1$ (see Fig. 2 (4)), then we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_2$, a contradiction.

Next we will show that $N_G(v) \cap U_1 = \emptyset$. Otherwise, there exists some vertex in U_1 adjacent to v , say $v \sim x_1$. Recall that $v \sim y_1$ (see Fig. 2 (5)) according to the above arguments, we get the induced subgraph $G[v_1, x_1, y_1, v_4, v] = I_5$, a contradiction.

Summarizing the above discussion, we know that $V(G') \cup \{v\}$ induces a subgraph $J(a+1, b)$ of G . This is impossible since $G' = J(a, b)$ is assumed to be the maximal induced subgraph including $J(2, 1)$.

Case 2. $v \sim v_4$ and $v \not\sim v_1$;

As similar as Case 1, by symmetry we can also deduce that $G[V(G') \cup \{v\}] = J(a, b+1)$. This is also impossible.

We complete this proof. \square

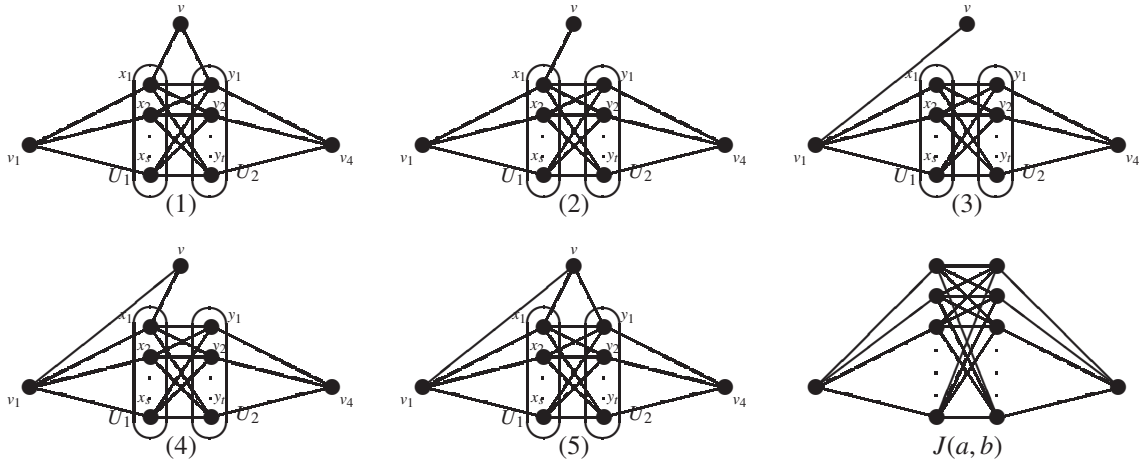


Figure 2: The graphs used in Lemma 3.6

After the completion of the preparations, we get one of our main results.

Theorem 3.1. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then $d(G) = 2$.*

Proof. By Lemma 3.1 and Remark 1, we get that G is P_5 -free and $d(G) = 2$ or $d(G) = 3$. Assume by contradiction that $d(G) = 3$. Let $d(v_1, v_4) = 3$ and $P = v_1 v_2 v_3 v_4$ a shortest path between v_1 and v_4 . By Lemma 3.4, G contains at least one of I_i (labelled as Fig. 1) as an induced subgraph for $i = 1, 2, \dots, 5$.

Suppose that I_1 is an induced subgraph of G . Note that $d_G(v_4, u) = d_{I_1}(v_4, u) - 1 = 2$ or $d_G(v_4, u) = d_{I_1}(v_4, u) = 3$. We get that either M_1 or M'_1 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_1 , where

$$M_1 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -2 \\ -2 & -1 & -2 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}, M'_1 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -3 \\ -2 & -1 & -2 & -3 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

If M_1 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, x_3, x_4, 0)$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$ such that $M_1 x = \partial_1^L x$. Consider the fifth

entry of both sides of $M_1x = \partial_1^Lx$, we have $-2x_1 - x_2 - 2x_3 - 2x_4 = 0$. It follows that $x_2 = 0$ and $x_1 + x_3 + x_4 = 0$. Next we consider the second entry of both sides of $M_1x = \partial_1^Lx$, we have $-x_1 - x_3 - 2x_4 = 0$. It follows that $x_4 = 0$ and $x_1 + x_3 = 0$. We consider the fourth entry of both sides of $M_1x = \partial_1^Lx$, we have $-3x_1 - x_3 = 0$. It follows that $x_1 = x_3 = 0$. Thus, we have $x = 0$, a contradiction. If M'_1 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $y = (0, y_2, y_3, y_4, y_5)^T$ satisfying

$$y_2 + y_3 + y_4 + y_5 = 0 \quad (1)$$

such that

$$M'_1y = \partial_1^Ly. \quad (2)$$

Consider the first entry of both sides of Eq. (2), we have

$$-y_2 - 2y_3 - 3y_4 - 2y_5 = 0. \quad (3)$$

Combining (1) and (3), we have $y_2 = y_4$. If $y_2 = y_4 = 0$, we consider the fourth entry of both sides of (2) and we get that $y_5 = 0$. It follows that $y = 0$, a contradiction. If $y_2 = y_4 \neq 0$, we consider the second entry of both sides of (2) and we get that $\partial_1^L = t_2 - \frac{y_3 + 2y_4 + y_5}{y_2} = t_2 - \frac{y_2 + y_3 + y_4 + y_5}{y_2}$. From (1), we have $\partial_1^L = t_2$. It contradicts Corollary 3.1.

Suppose that I_2 is an induced subgraph of G . Note that $d_G(v_4, u) = d_{I_2}(v_4, u) = 3$ or $d_G(v_4, u) = d_{I_2}(v_4, u) - 1 = 2$. We get that the matrix M_2 or M'_2 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_2 , where

$$M_2 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -3 \\ -1 & -1 & -2 & -3 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}, M'_2 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -1 & -2 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

If M_2 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_2x = \partial_1^Lx$. We successively consider the second, the fourth and the third entries of both sides of $M_2x = \partial_1^Lx$, we get that $x_3 = x_4 = 0$ and $x_1 + x_5 = 0$. If $x_1 = 0$, then $x_5 = 0$ and $x = 0$, a contradiction. If $x_1 \neq 0$, consider the first entry of both sides of $M_2x = \partial_1^Lx$, we get that $\partial_1^L = t_1 - \frac{x_5}{x_1} = t_1 + 1$. It contradicts Corollary 3.1. If M'_2 is a principal submatrix of $\mathcal{D}^L(G)$, by Lemma 3.5, there exists an eigenvector $y = (y_1, y_2, y_3, y_4, 0)^T$ satisfying $y_1 + y_2 + y_3 + y_4 = 0$ such that $M'_2y = \partial_1^Ly$. Consider the fifth entry of both sides of $M'_2y = \partial_1^Ly$, we have $-y_1 - y_2 - 2y_3 - 2y_4 = 0$. It leads to that $y_1 + y_2 = y_3 + y_4 = 0$. If $y_3 = y_4 = 0$, we consider the third entry of both sides of $M'_2y = \partial_1^Ly$ and we get that $y_1 = y_2 = 0$. It leads to that $y = 0$, a contradiction. If $y_1 = y_2 = 0$, we consider the second entry of both sides of $M'_2y = \partial_1^Ly$ and we get that $y_3 = y_4 = 0$. It leads to that $y = 0$, a contradiction. If $y_1, y_2, y_3, y_4 \neq 0$, without loss of generality, we may suppose that $y = (a, -a, 1, -1, 0)$. Consider the third entry of both sides of $M'_2y = \partial_1^Ly$, we have $\partial_1^L = t_3 + 1 - a$. By Corollary 3.1, we have $a < 0$. Consider the fourth entry of both sides of $M'_2y = \partial_1^Ly$, we have $\partial_1^L = t_4 + 1 + a$. By Corollary 3.1, we have $a > 0$, a contradiction.

Suppose that I_4 is an induced subgraph of G . We get that the matrix M_4 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_4 , where

$$M_4 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -2 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -2 & -1 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_4 x = \partial_1^L x$. Consider the second and the fourth entries of both sides of $M_4 x = \partial_1^L x$ successively, we get that $x_4 = 0$, $x_1 = x_3$ and $x_5 = -2x_1$. If $x_1 = x_3 = 0$, then $x = 0$, a contradiction. If $x_1 = x_3 \neq 0$, consider the third entry of both sides of $M_4 x = \partial_1^L x$ and we get that $\partial_1^L = t_3$. It contradicts Corollary 3.1.

Suppose that I_5 is an induced subgraph of G . We get that the matrix M_5 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_5 , where

$$M_5 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -1 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, 0, x_3, x_4, x_5)^T$ satisfying $x_1 + x_3 + x_4 + x_5 = 0$ such that $M_5 x = \partial_1^L x$. We successively consider the second and the fourth entries of both sides of $M_5 x = \partial_1^L x$, we have that $x_4 = 0$, $x_1 = x_3$ and $x_5 = -2x_3$. If $x_3 = 0$, we have $x = 0$, a contradiction. If $x_3 \neq 0$, consider the third entry of both sides of $M_5 x = \partial_1^L x$, we have $\partial_1^L = t_3$. It contradicts Corollary 3.1.

Suppose that I_3 is an induced subgraph of G . On the one hand, we get that the matrix M_3 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to I_3 , where

$$M_3 = \begin{pmatrix} t_1 & -1 & -2 & -3 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -1 \\ -3 & -2 & -1 & t_4 & -2 \\ -1 & -2 & -1 & -2 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, 0, x_4, x_5)^T$ satisfying $x_1 + x_2 + x_4 + x_5 = 0$ such that $M_3 x = \partial_1^L x$. We successively consider the third, the first and the fourth entries of both sides of $M_3 x = \partial_1^L x$, then we get that $x_1 = x_4 = 0$ and $x_2 + x_5 = 0$. If $x_2 = x_5 = 0$, then $x = 0$, a contradiction. If $x_2 \neq 0$, without loss of generality, we may suppose that $x = (0, 1, 0, 0, -1)^T$. Consider the second entry of $M_3 x = \partial_1^L x$, we get that $\partial_1^L = t_2 + 2$. By Corollary 3.1, we get that

$$Tr(v_2) = \max_{v \in V(G)} Tr(v). \quad (4)$$

On the other hand, recall that G is P_5 -free. Moreover, by the arguments above, we have that G contains no induced I_1 , I_2 , I_4 or I_5 . Therefore, by Lemma 3.6, we have that $G =$

$J(a, b)$ with roots v_1 and v_4 . By simple calculation, we have $Tr(v_1) = a + 2b + 3$, $Tr(v_4) = 2a + b + 3$, $Tr(x) = 2a + b + 1$ for every $x \in N(v_1)$ and $Tr(y) = 2b + a + 1$ for every $y \in N(v_4)$. Note that $v_2 \in N(v_1)$. We get that

$$Tr(v_2) = 2a + b + 1 < 2a + b + 3 = Tr(v_4),$$

which contradicts (4).

We complete the proof. \square

The result above showed that the graphs in $\mathcal{G}(n, n - 3)$ have diameter 2. In fact, we can further obtain that G is the join of two graphs. To prove this, we need the following result.

Lemma 3.7. *Let $G \in \mathcal{G}(n, n - 3)$ with $n \geq 6$, then none of $J_1 (= C_5)$, J_2 or J_3 (shown in Fig. 3) can be an induced subgraph of G .*

Proof. Assume by contradiction that $J_1 = C_5$ is an induced subgraph of G . We get that the matrix N_1 is a principal submatrix of $\mathcal{D}^L(G)$ with respect to J_1 , where

$$N_1 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{matrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (0, x_2, x_3, x_4, x_5)^T$ satisfying $x_2 + x_3 + x_4 + x_5 = 0$ such that $N_1 x = \partial_1^L x$. From the first entry of $N_1 x = \partial_1^L x$, we have $-x_2 - 2x_3 - 2x_4 - x_5 = 0$. Therefore, we have $x_3 + x_4 = 0$ and $x_2 + x_5 = 0$. If $x_3 = x_4 = 0$, consider the third entry of both sides of $N_1 x = \partial_1^L x$ and we get that $x_2 = x_5 = 0$. It leads to that $x = 0$, a contradiction. If $x_2 = x_5 = 0$, consider the second entry of both sides of $N_1 x = \partial_1^L x$ and we get that $x_3 = x_4 = 0$. It leads to that $x = 0$, a contradiction. If $x_2, x_3, x_4, x_5 \neq 0$, without loss of generality, we may suppose that $x = (0, a, 1, -1, -a)^T$. Thus, we have

$$\begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ 1 \\ -1 \\ -a \end{pmatrix} = \partial_1^L \begin{pmatrix} 0 \\ a \\ 1 \\ -1 \\ -a \end{pmatrix}. \quad (5)$$

Consider the fourth entry of both sides of (5), we have

$$\partial_1^L = t_4 + a + 1.$$

By Corollary 3.1, we have $a \geq 1$. Consider the fifth entry of both sides of Eq. (5), we have

$$\partial_1^L = t_5 + \frac{1}{a} + 2.$$

By Lemma 3.3, we get that ∂_1^L is integral. Therefore, a and $\frac{1}{a}$ are both integral. Thus, we have $a = 1$ and $\partial_1^L = t_4 + 2 = t_5 + 3$. It follows that

$$t_4 = t_5 + 1. \quad (6)$$

On the other hand, by Lemma 3.5, there also exists an eigenvector $y = (y_1, 0, y_3, y_4, y_5)^T$ satisfying $y_1 + y_3 + y_4 + y_5 = 0$ such that $N_1 y = \partial_1^L y$. From the second entry of $N_1 y = \partial_1^L y$, we have $-y_1 - y_3 - 2y_4 - 2y_5 = 0$. Therefore, we have $y_4 + y_5 = 0$ and $y_1 + y_3 = 0$. If $y_1 = y_3 = 0$ or $y_4 = y_5 = 0$, we also get $y = 0$, a contradiction. If $y_1, y_3, y_4, y_5 \neq 0$, without loss of generality, we may suppose that $y = (b, 0, -b, 1, -1)^T$. Thus, we have

$$\begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -2 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -2 & -2 & -1 & t_5 \end{pmatrix} \begin{pmatrix} b \\ 0 \\ -b \\ 1 \\ -1 \end{pmatrix} = \partial_1^L \begin{pmatrix} b \\ 0 \\ -b \\ 1 \\ -1 \end{pmatrix}. \quad (7)$$

Consider the fourth and the fifth entries of both sides of Eq. (7), we have

$$\partial_1^L = t_4 - b + 1 = t_5 - b + 1.$$

It follows that $t_4 = t_5$, which contradicts (6).

Assume by contradiction that J_2 is an induced subgraph of G . We get that the matrix N_2 is a principal submatrix of \mathcal{D}^L with respect to J_2 , where

$$N_2 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -2 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -1 & -2 & -1 & t_5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{pmatrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, x_3, x_4, 0)^T$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$ such that $N_2 x = \partial_1^L x$. We successively consider the fifth, the third, the first and the fourth entries of both sides of $N_2 x = \partial_1^L x$, then we get that $x = 0$, a contradiction.

Assume by contradiction that J_3 is an induced subgraph of G . We get that the matrix N_3 is a principal submatrix of \mathcal{D}^L with respect to J_3 , where

$$N_3 = \begin{pmatrix} t_1 & -1 & -2 & -2 & -1 \\ -1 & t_2 & -1 & -2 & -1 \\ -2 & -1 & t_3 & -1 & -1 \\ -2 & -2 & -1 & t_4 & -1 \\ -1 & -1 & -1 & -1 & t_5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ u \end{pmatrix}.$$

By Lemma 3.5, there exists an eigenvector $x = (x_1, x_2, 0, x_4, x_5)^T$ satisfying $x_1 + x_2 + x_4 + x_5 = 0$ such that $N_3 x = \partial_1^L x$. Consider the third, the first, the fourth and the second entries of both sides of $N_3 x = \partial_1^L x$ successively, we get that $x = 0$, a contradiction. \square

Using the above tools, we get the following result.

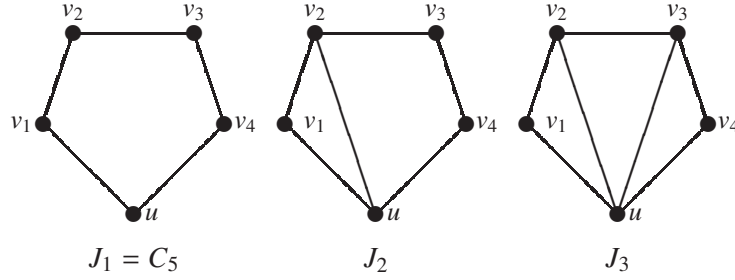


Figure 3: The graphs in Lemma 3.7

Theorem 3.2. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 6$, then \bar{G} is disconnected. It means that G is the join of some connected graphs.*

Proof. By Lemma 2.2, it suffices to show that G contains no induced P_4 . Assume by contradiction that G contains an induced $P_4 = v_1 v_2 v_3 v_4$. By Theorem 3.1, we have $d(G) = 2$. Therefore, there exists a vertex $u \in V(G)$ such that $u \sim v_1, v_4$. It follows that at least one of J_1, J_2 and J_3 will be an induced subgraph of G , contradicts Lemma 3.7. \square

For any graph $G \in \mathcal{G}(n, n-3)$, we see that G has at most four distinct eigenvalues, and we also have $\partial_{n-1}^L(G) = n$ by Theorems 2.3 and 3.2. Denote by

$$\mathcal{H}_1(n) = \{G \in \mathcal{G}(n, n-3) \mid \text{Spec}_{\mathcal{L}}(G) = [(\partial_1^L)^{n-3}, \partial_{n-2}^L, \partial_{n-1}^L = n, \partial_n^L = 0]\},$$

and

$$\mathcal{H}_2(n) = \{G \in \mathcal{G}(n, n-3) \mid \text{Spec}_{\mathcal{L}}(G) = [(\partial_1^L)^{n-3}, \partial_{n-2}^L = \partial_{n-1}^L = n, \partial_n^L = 0]\}.$$

Therefore, $\mathcal{H}_1(n)$ and $\mathcal{H}_2(n)$ are the sets of graphs with four and three distinct eigenvalues in $\mathcal{G}(n, n-3)$, respectively. Thus we have the disjoint decomposition

$$\mathcal{G}(n, n-3) = \mathcal{H}_1(n) \cup \mathcal{H}_2(n).$$

Mohammadian [9] gave the following result.

Lemma 3.8 ([9], Theorem 8). *Let G be a graph on $n \geq 5$ vertices whose distinct Laplacian eigenvalues are $0 < \alpha < \beta < \gamma$. Then the multiplicity of α is $n-3$ if and only if G is one of the graphs $K_{2,n-2}$, $K_{n/2,n/2} + e$ or $K_{1,n-1} + e$, where $K_{n/2,n/2} + e$ and $K_{1,n-1} + e$ are the graphs obtained from $K_{n/2,n/2}$ and $K_{1,n-1}$, respectively, by adding an edge e joining any two non-adjacent vertices.*

Note that, when $d(G) = 2$, there exists a correspondence between the distance Laplacian spectrum and the Laplacian spectrum of G . We have the following result.

Corollary 3.2. *For an integer $n \geq 6$, we have $\mathcal{H}_1(n) = \{K_{2,n-2}, K_{n/2,n/2} + e, K_{1,n-1} + e\}$, and their distance Laplacian spectra are given by*

$$\begin{cases} \text{Spec}_{\mathcal{L}}(K_{2,n-2}) = \{(2n-2)^{n-3}, n+2, n, 0\} \\ \text{Spec}_{\mathcal{L}}(K_{n/2,n/2} + e) = \{(\frac{3n}{2})^{n-3}, \frac{3n}{2} - 2, n, 0\} \\ \text{Spec}_{\mathcal{L}}(K_{1,n-1} + e) = \{(2n-1)^{n-3}, 2n-3, n, 0\} \end{cases} \quad (8)$$

Proof. Let $G \in \mathcal{H}_1(n)$ and $\text{Spec}_{\mathcal{L}}(G) = \{(\partial_1^L)^{n-3}, \partial_{n-2}^L, n, 0\}$ where $\partial_1^L > \partial_{n-2}^L > n$. By Theorem 3.1, we have $d(G) = 2$. Therefore, by Theorem 2.2, the Laplacian spectrum of G is $\{n, 2n - \partial_{n-2}^L, (2n - \partial_1^L)^{n-3}, 0\}$. Thus, we get that $G \in \{K_{2,n-2}, K_{n/2,n/2} + e, K_{1,n-1} + e\}$ from Lemma 3.8. Conversely, note that all of $K_{2,n-2}$, $K_{n/2,n/2} + e$ and $K_{1,n-1} + e$ are the join of two graphs, by Lemma 2.1 (iv) and Theorem 2.2, we obtain their distance Laplacian spectra, which are shown in (8). Therefore, $K_{2,n-2}, K_{n/2,n/2} + e, K_{1,n-1} + e \in \mathcal{H}_1(n)$, and the result follows. \square

In what follows we characterise $\mathcal{H}_2(n)$.

Lemma 3.9. *For an integer $n \geq 6$, we have $\mathcal{H}_2(n) = \{K_2 \nabla(n-2)K_1, K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}, K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}\}$, and their distance Laplacian spectra are given by*

$$\begin{cases} \text{Spec}_{\mathcal{L}}(K_2 \nabla(n-2)K_1) = \{(2n-2)^{n-3}, n^2, 0\} \\ \text{Spec}_{\mathcal{L}}(K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}) = \{((3n-1)/2)^{n-3}, n^2, 0\} \\ \text{Spec}_{\mathcal{L}}(K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}) = \{(4n/3)^{n-3}, n^2, 0\} \end{cases} \quad (9)$$

Proof. Let $G \in \mathcal{H}_2(n)$ and $\text{Spec}_{\mathcal{L}}(G) = \{(\partial_1^L)^{n-3}, n^2, 0\}$ where $\partial_1^L > n$. By Theorem 3.1, we get that $d(G) = 2$. Therefore, by Theorem 2.2, the Laplacian spectrum of G is $\{n^2, (2n - \partial_1^L)^{n-3}, 0\}$. By Lemma 2.1 (iii), the Laplacian spectrum of \bar{G} is $\{(\partial_1^L - n)^{n-3}, 0^3\}$. By Lemma 2.1 (i), \bar{G} has exactly three components, denoted by G_1, G_2 and G_3 . Moreover, by Lemma 2.1 (ii), G_1, G_2 and G_3 are either complete graphs of the same order or isolate vertices. If none of them is an isolate vertex, then $G_1 \cong G_2 \cong G_3 \cong K_{n/3}$. It follows that $G = \overline{3K_{n/3}} = K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}$. If there's exactly one of them is an isolate vertex, say G_3 , then $G_1 \cong G_2 \cong K_{(n-1)/2}$. It follows that $G = \overline{2K_{(n-1)/2} \cup K_1} = K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}$. If there are exactly two of them are isolate vertices, say G_2 and G_3 , then $G_1 \cong K_{n-2}$. It follows that $G = \overline{K_{n-2} \cup 2K_1} = K_2 \nabla(n-2)K_1$. Conversely, note that all of $K_2 \nabla(n-2)K_1, K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}$ and $K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}$ are the join of two graphs, by Lemma 2.1 (iv) and Theorem 2.2, we obtain their distance Laplacian spectra, which are shown in (9). Therefore, $K_2 \nabla(n-2)K_1, K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}, K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}} \in \mathcal{H}_2(n)$, and the result follows. \square

Recall that $\mathcal{G}(n, n-3) = \mathcal{H}_1(n) \cup \mathcal{H}_2(n)$. Combining Corollary 3.2 and Lemma 3.9, we completely determine $\mathcal{G}(n, n-3)$ in the following result.

Theorem 3.3. *For an integer $n \geq 6$, we have*

$$\mathcal{G}(n, n-3) = \{K_{2,n-2}, K_{1,n-1} + e, K_{n/2,n/2} + e, K_2 \nabla(n-2)K_1, K_1 \nabla K_{\frac{n-1}{2}, \frac{n-1}{2}}, K_{\frac{n}{3}, \frac{n}{3}, \frac{n}{3}}\}.$$

Remark 2. *By using the software SageMath, we get the graphs with $m(\partial_1^L) = n-3$ for $n=4$ and $n=5$. That is,*

$$\begin{cases} \mathcal{G}(4, 1) = \{P_4, K_{1,3} + e, K_2 \nabla 2K_1\} \\ \mathcal{G}(5, 2) = \{K_{2,3}, K_{1,4} + e, K_2 \nabla 3K_1, K_1 \nabla K_{2,2}, C_5\} \end{cases}.$$

We end up this paper by the following result.

Corollary 3.3. *Let $G \in \mathcal{G}(n, n-3)$ with $n \geq 5$ then G is determined by its distance Laplacian spectrum.*

Proof. Let $H \in \mathcal{G}(n)$ with $\text{Spec}_{\mathcal{L}}(H) = \text{Spec}_{\mathcal{L}}(G)$. We get that $H \in \mathcal{G}(n, n-3)$. Then, the result follows by pairwise comparing the distance Laplacian spectra of graphs in $\mathcal{G}(n, n-3)$, which are presented in (8) and (9). \square

References

- [1] M. Aouchiche, P. Hansen, Distance spectra of graphs: A survey, *Linear Algebra Appl.* 458 (2014) 301–384.
- [2] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.* 439 (2013) 21–33.
- [3] M. Aouchiche, P. Hansen, A Laplacian for the distance matrix of a graph, *Czechoslovak Mathematical Journal* 64 (2014) 751–761.
- [4] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Uni. Press, New York, 2010.
- [5] D. Corneil, H. Lerchs, L. Burlingham. Complement reducible graphs, *Discrete Appl. Math.* 3 (1981) 163–174.
- [6] Celso M. Da Silva JR., et al., A note on a conjecture for the distance Laplacian matrix, *Electr. J. Linear Algebra* 31 (2016) 60–68.
- [7] van Dam, E.R., Haemers, W.H., Graphs with constant μ and $\bar{\mu}$, *Discrete Math.* 182 (1998) 293–307.
- [8] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, New York, 1992.
- [9] A. Mohammadian, B. Tayfeh-Rezaie, Graphs with four distinct Laplacian eigenvalues, *J. Algebra Comb.* 34 (2011) 671–682.
- [10] M. Nath, S. Paul, On the distance Laplacian spectra of graphs, *Linear Algebra Appl.* 460 (2014) 97–110.
- [11] F. Tian, D. Wong, J. Rou, Proof for four conjectures about the distance Laplacian and distance signless Laplacian eigenvalues of a graph, *Linear Algebra Appl.* 471 (2015) 10–20.